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Notes and Comment

Evaluating the importance of the convex hull in solving the Euclidean version of the traveling salesperson problem: Reply to Lee and Vickers (2000)

JAMES N. MACGREGOR
University of Victoria,
Victoria, British Columbia, Canada

and

THOMAS C. ORMEROD
Lancaster University, Lancaster, England

Lee and Vickers (2000) suggest that the results of MacGregor and Ormerod (1996), showing that the response uncertainty to traveling salesperson problems (TSPs) increases with increasing numbers of nonboundary points, may have resulted as an artifact of constraints imposed in the construction of stimuli. The fact that similar patterns of results have been obtained for our "constrained" stimuli, for a stimulus constructed under different constraints, for 13 randomly generated stimuli, and for random and patterned 48-point problems provides empirical evidence that the results are not artifactual. Lee and Vickers further suggest that, even if not artifactual, the results are in principle limited to arrays of fewer than 50 points and that, beyond this, the total number of points and number of nonboundary points are "diagnostically equivalent." This claim seems to us incorrect, since arrays of any size can be constructed that will permit experimental tests of whether problem difficulty is influenced by the number of nonboundary points, or the total number of points, or both. We present a reanalysis of our original data using hierarchical regression analysis which indicates that both factors may influence problem complexity.

The Euclidean form of the traveling salesperson problem (TSP) consists of finding the shortest path passing through a set of points in the plane and returning to the starting point. The results from two experiments reported by us (MacGregor & Ormerod, 1996) indicated that human subjects were relatively proficient at solving 10- and 20-point TSPs, that in generating solutions they were guided by the boundary points of the problems, and that the difficulty of these TSPs for human subjects was determined by the number of nonboundary points. In their comment, Lee and Vickers (2000) question the generality and/or validity of the last two conclusions on the grounds of the stimuli used, proposing "some significant limitations . . . and one important qualification" (p. 227).

Correspondence should be addressed to J. MacGregor, School of public Administration, University of Victoria, P. O. Box 1700, Victoria, BC, V8W 2Y2 Canada (e-mail: jmacgreg@uvic.ca).

The TSPs in our original experiments were designed for a particular purpose. In the first experiment, we used 10-point problems ranging in number of interior points from 1 to 6. We believed that boundary points would not add to the complexity of TSPs for human subjects, and we further suspected that points falling close to the boundary might act effectively like boundary points. Because of this we took steps to ensure that all of the interior points were clearly separated from the boundary. This concern—that problems whose interior points were close to the boundary would be simpler than those with more embedded interior points—has since received empirical support (MacGregor, Ormerod, & Chronicle, 1999). In our second experiment, we used seven 20-point problems ranging from 4 to 16 interior points. Because we suspected that proximity between interior points would affect solutions, we took additional steps to ensure that points did not become crowded together in the centers of the figures. This concern has also been justified empirically (MacGregor & Ormerod, 1996; MacGregor et al., 1999). Therefore, we did not, as Lee and Vickers suggest, sacrifice generality for no benefit, but in an effort to control for other potentially relevant factors.

Lee and Vickers (2000) argue that the constraints that we imposed in constructing the stimuli (1) may limit the generality of the findings, and/or (2) may have generated them as an artifact. In support of (1), Lee and Vickers demonstrate that the distribution of points in our stimuli would have been highly unlikely to have occurred if the problems had been generated randomly. This is no doubt the case, since we intentionally constructed our stimuli to test a specific hypothesis while controlling for other factors, rather than generated them randomly. The question of whether the results generalize to other TSPs, constructed on different principles (including random generation), is very important. In fact, we discussed this issue in the original article. We also specifically addressed the issue of artifactual results raised again now by Lee and Vickers. They suggest that, given the generally circular forms of the arrays we used, it "is difficult to believe that subjects would not be sensitive to the high degree of rotational symmetry" (p. 227). This seems to echo our expressed concern that the circular form of the problems used might "implicitly have prompted subjects to pay attention to the boundary of the convex hull" (MacGregor & Ormerod, 1996, p. 537). However, at the same time, we also provided two sources of evidence indicating that the results were not artifactual and that they would generalize to other problems. First, we used a problem from the operations research literature that was not subject to the same construction constraints as were our own, reported by Barachet (1957) and used by others to test solution approaches to TSPs (Dantzig, Fulkerson, & Johnson, 1959). The results for this problem were consistent with those for our own "constrained" problems, and subjects continued to exhibit the same performance characteristics. Second, we reanalyzed previously unpublished data from 13 TSPs ranging from 10 to 60 nodes, which were generated uniformly at random from an array that allowed the resulting problems to be plotted on standard 8.5 × 11 sheets. Again, the pattern of results was consistent with those obtained with our constrained TSPs. The evidence that we presented in the original article therefore indicated the same performance characteristics with (1) constrained TSPs, (2) a 10-node TSP constructed on other principles, and (3) 13 random TSPs. More recently, we have used two 48-point problems, one randomly generated, one highly patterned. In both cases, the results confirmed the previous results, that subjects produce relatively good solutions and that boundary points guide the solution process (MacGregor et al., 1999). Lee and Vickers's claim that our results are an artifact does not therefore appear to be supported by the available evidence.

Lee and Vickers (2000) further propose that even if our results were not an artifact, the influence of the number of interior points is limited in principle to TSPs of less than 50 points or so. They base this conclusion on computer simulations in which arrays varying in size from 10 to 100 points were randomly generated within the unit square. They observed that the mean and variance of the number of boundary points appears to asymptote, so that for problems of around 50 points or greater "the concept 'total number of points' is diagnostically equivalent to the concept 'number of nonboundary points,' in the sense that they effectively differ only by a constant" (p. 228). If by the term "diagnostic equivalence" Lee and Vickers mean that the numbers of total points and nonboundary points will be correlated in large, randomly generated TSPs, then we agree. However, even if they are correlated in randomly generated arrays, it remains possible and meaningful to distinguish between the two variables. Clearly it is possible to construct problems of any desired size in which total number of points and number of nonboundary points vary orthogonally, thereby allowing both factors to be tested independently. We feel confident that if experiments were conducted comparing TSPs of, say, 200 points having no or few interior points with TSPs of 100 points having, say, 60 or more interior points, the former would be much simpler than the latter. However, until such experiments are conducted, it may be rash to draw conclusions. Experiments that distinguish between the effects of total points and interior points could also be conducted with randomly generated stimuli, if there is a concern about using specially constructed stimuli. For example, this could be done by selecting problems from the upper and lower ranges of the distributions of nonboundary points shown in Lee and Vickers's Figure 1.

In their closing remarks, Lee and Vickers (2000) cite results in which they found the optimality of solutions to random arrays to be "independent of the number of boundary points, although . . . the total number of points did influence solutions' speed and response uncertainty" (p. 228). If Lee and Vickers are correct in stating that the number of boundary points becomes effectively a constant in large random arrays, then the result must therefore mean that *both* the number of points *and* the number of nonboundary points were correlated with performance, since the two will differ approximately by a constant. This is consistent with what the MacGregor/Ormerod hypothesis predicts: that performance varies with the number of nonboundary points.

Finally, we should point to a real limitation of our study that Lee and Vickers (2000) did not discuss. In our original experiments, the number of nonboundary points was varied while the total number of points was held constant (at 10 and 20 points for the two experiments, respectively). The results indicated that problem complexity increases with the number of nonboundary points for arrays of a fixed size. However, the results did not indicate whether, *in addition*, complexity varies also with total number of points, independently of the number of interior points. To shed some light on this, we reanalyzed our original data, combining the results from both experiments. Using hierarchical regression analysis with response uncertainty as the dependent variable and total points and interior points as independent variables (entered in that order), we obtained the results in the upper half of Table 1. They show that the initial effect of total number of points was significant and substantial, accounting for 66% of the variance in response uncertainty [$F(1,11) = 21.65$, $p = .001$]. Entering the number of interior points next increased R^2 by .21 to .87, a significant increase [change in $F(1,10) = 15.50$, $p = .003$]. The analysis suggests that both total points and number of

Table 1
Hierarchical Regression Analysis of Response Uncertainty as a Function of Total Number of Points and Number of Interior Points

Independent Variable	R	R^2	*SE* of Estimate	R^2 Change	*F* Change	*df*(1)	*df*(2)	Sig. *F* Change
Analysis 1								
Total points	.81	.66	3.00	.66	21.65	1	11	.001
Interior points	.93	.87	1.97	.21	15.50	1	10	.003
Analysis 2								
Interior points	.90	.81	2.26	.81	46.39	1	11	.000
Total points	.93	.87	1.97	.06	4.50	1	10	.06

nonboundary points are important factors in problem difficulty. However, the results are somewhat different if the number of interior points is the first variable to be entered in the regression analysis. These results are shown in the lower part of Table 1. When entered first, the number of interior points accounts for 81% of the total variance, and the subsequent inclusion of the total points variable just fails to add significantly, with an increase in R^2 of .06 [change in $F(1,10) = 4.50$, $p = .06$]. The ambiguity in these findings due to the order of entry of the variables is inherent in regression analysis unless the independent variables are orthogonal. However, these preliminary results provide some indication that the total number of points and the number of interior points may make significant independent contributions to the complexity of TSPs. In this regard, Lee and Vickers (2000) and MacGregor and Ormerod (1996) may both be partially right. Clearly, there is a need to clarify the mutual roles of these factors, and experiments to do so are in preparation.

In closing, we should emphasize that our experiments have been limited to the Euclidean version of the problem, which is simpler mathematically than the general TSP. As we said in our original article (MacGregor & Ormerod, 1996), we expect that the generally good quality of human solutions that we observed will be limited to problems presented spatially in the plane and will not generalize to other representations of the problem.

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41st Annual Meeting of the Psychonomic Society
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The 41st Annual Meeting of the Psychonomic Society will be held in New Orleans, Louisiana, at the Hyatt Regency Hotel. Registration will begin on Thursday evening, November 16. A poster session will be held on Thursday evening; spoken sessions will begin on the morning of Friday, November 17. Sessions will continue through noon on Sunday, November 19.

Programs and hotel reservation cards were mailed to members and associates in the beginning of September. Additional programs will be available at the meeting registration desk for $10.00.

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